Inclusion and exclusion in team semantics

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Introduction

We consider **exclusion** and **inclusion** atoms with different syntactic definitions and the axiomatizations of their **implication problems**.

The implication problems for atoms:
Does a set of atoms imply a given atom or not?

We also briefly discuss the team-based propositional logics obtained by adding these atoms.

Exclusion atoms

A team $T = \{s_1, \ldots, s_n\}$ is a finite set of assignments $s_i : \mathcal{V} \longrightarrow M$, where $\mathcal{V}$ is a set of variables and M is a set of values.
For $x = \langle x_1, \ldots, x_n \rangle$, we write $s(x)$ for $\langle s(x_1), \ldots, s(x_n) \rangle$.

Exclusion atoms $x|y$ (with $|x| = |y|$)
$T \models x|y$ if and only if for all $s_1, s_2 \in T$, $s_1(x) \neq s_2(y)$.

	x_1	x_2	y_1	y_2
s_1	▲	◇	◇	▲
s_2	▲	◇	◇	▲
s_3	◇	◇	◇	▲

	x_1	x_2	y_1	y_2
s_1	▲	◇	◇	▲
s_2	▲	◇	◇	◇
s_3	◇	◇	◇	▲

It follows that exclusion atoms are *downward closed*.

Systems for exclusion atoms

With “repetitions” we mean repeated variables within one sequence.

(E1-E4) Rules for repetition free exclusion [Cf. Casanova, Vidal ’83]

(E1) If $x|x$, then $y|z$

(E2) If $x|y$, then $y|x$

(E3) If $x|y$, then $xu|yv$

(E4) If $xyz|uvw$, then $xzy|uwv^*$ $^*(|x| = |u|$ and $|y| = |v|)$

(E1-E6) Rules for exclusion [H. ’24]

(E5) If $xuu|yvv$, then $xu|yv$

(E6) If $xw|yw$, then $zz|xy$.

Typical completeness proof

Completeness theorem

Let $\Sigma \cup \{x|y\}$ be a set of exclusion atoms. If $\Sigma \models x|y$ then $\Sigma \vdash_{\{E1-E6\}} x|y$.

We prove the contrapositive (if $\Sigma \nvdash x|y$, then $\Sigma \not\models x|y$) using a counterexample team.

	x_1	$\dots$	x_n	y_1	$\dots$	y_n	z_1	z_2	$\dots$
s_1	a_1	$-$	a_n		$\dots$				$\dots$
s_2		$\dots$		a_1	$-$	a_n			$\dots$

The team is filled with values that appear only once, except for one repetition of $\langle a_1, \dots, a_n \rangle$ such that $s_1(x) = s_2(y)$.

We then need to check that $T \not\models x|y$, and that $T \models \Sigma$.

Approximate exclusion

Approximate exclusion
Let p be a real number such that $0 \leq p \leq 1$. $T \models x|_p y$ if and only if there is a subteam $T' \subseteq T$, $|T'| \leq p \cdot |T|$, such that $T \setminus T' \models x|y$.

Approximate exclusion atoms (e.g., $x|_{\frac{1}{100}} y$) are suitable when we accept that the data set is flawed, up to some degree.

	x_1	x_2	y_1	y_2
s_1	●	●	▲	▲
s_2	●	♠	▲	♢
⋮	⋮	⋮	⋮	⋮
s_{98}	●	♠	▲	♢
s_{99}	♠	●	♠	♠
s_{100}	♠	♠	♢	▲

System for approximate exclusion

(A) Rules for approximate exclusion [H. '24]

(A1) For $p < 1$, if $x|_p x$, then $y|_0 z$

(A2) If $x|_p y$, then $y|_p x$

(A3) If $x|_p y$, then $xu|_p yv$

(A4) If $xuu|_p yvv$, then $xu|_p yv$

(A5) If $xyz|_p uvw$, then $xzy|_p uwv^*$

(A6) If $xw|_p yw$, then $zz|_p xy$

(A7) For $q \leq p \leq 1$, if $x|_q y$, then $x|_p y$

(A8) $x|_1 y$.

$^*(|x| = |u|$ and $|y| = |v|)$

Completeness theorem

Let $\Sigma \cup \{x|_p y\}$ be a set of approximate exclusion atoms with $0 \leq p < \frac{1}{2}$ such that if there are $u|_q v \in \Sigma$ with $q > p$, then $r = min\{q > p : u|_q v \in \Sigma\}$ exists.
If $\Sigma \models x|_p y$ then $\Sigma \vdash_A x|_p y$.

Again, we show that if $\Sigma \nvdash_A x|_p y$, then $\Sigma \not\models x|_p y$ using a counterexample team.

	x_1	$\dots$	x_n	y_1	$\dots$	y_n	z_1	z_2	$\dots$
s_1	a_1	$-$	a_n		$\dots$				$\dots$
s_2	b_1	$-$	b_n		$\dots$				$\dots$
s_3		$\dots$		a_1	$-$	a_n			$\dots$
s_4		$\dots$		b_1	$-$	b_n			$\dots$
s_5		$\dots$			$\dots$				$\dots$

We adjust the number of values that repeat and can add "good" lines like s_5 with values only appearing once.

Propositional exclusion atoms

We observe that the counterexample teams have many values.

Can we define a team that uses only two values to prove completeness of the system?

No!

Consider the following entailment, which is sound in the propositional setting, but not in the first-order setting: $x_1|x_2,\ x_2|x_3,\ x_3|x_1 \models q|z.$

	x_1	x_2	x_3	$z_1 \dots$
s_1	0	1	0	$\dots$
s_2	0	1	0	$\dots$

	x_1	x_2	x_3	$z_1 \dots$
s_1	0	1	2	$\dots$
s_2	0	1	2	$\dots$

Open problem: Axiomatization of propositional exclusion atoms.

First-order VS Propositional atoms

Atom	Same axiomatization?
Dependence	Yes (GV13)
Independece (non-conditional)	Yes (GV13)
Exclusion	No
Inclusion	Yes (We will see this soon)
Anonymity	No

Propositional exclusion logic

In progress: Axiomatization of propositional exclusion **logic**.

Expressivity result known through the following observations:

- $CPL(|)$ is downwards closed and has the empty team property.
- $CPL(Dep)$ is complete for all downwards closed team properties with the empty team. (Yang, Väänänen '16)

$\Longrightarrow$ $CPL(|) \leq CPL(Dep)$.

- $CPL(Dep) \equiv CPL(Constancy)$.
- We can capture constancy of p by the (extended) exclusion atom $p|\neg p$ (or $\top p|p\bot$).

$\Longrightarrow$ $CPL(|) \equiv CPL(Dep)$.

Inclusion atoms

Inclusion atoms are written as $x \subseteq y$, where $|x| = |y|$. We recall their semantics,

$T \models x \subseteq y$ iff for all $s_1 \in T$ there exists $s_2 \in T : s_1(x) = s_2(y)$.

	x_1	x_2	y_1	y_2
s_1	▲	▲	▲	◇
s_2	▲	◇	◇	◇
s_3	◇	◇	▲	▲

	x_1	x_2	y_1	y_2
s_1	▲	▲	▲	◇
s_2	▲	◇	▲	◇
s_3	◇	◇	▲	▲

It follows that inclusion atoms are *union closed*.

Inclusion variants

Repetition free inclusion	$p_1p_2 \subseteq q_1q_2$	$(x_1x_2 \subseteq y_1y_2)$
Inclusion	$p_1p_2 \subseteq q_1q_1$	$(x_1x_2 \subseteq y_1y_1)$
Propositional inclusion with constants	$\top p_2p_3 \subseteq q_1\bot q_1$	
Extended inclusion	$\alpha_1\alpha_2 \subseteq \beta_1\beta_2$	

Example

Consider the team $T = \{s_{2021}, s_{2022}, s_{2023}\}$, and let $\{p_a, p_b\}$ be such that $s_{2021}(p_a) = 1$ iff student a passed their exams in 2021, etc.

$T \models p_a \subseteq p_b$ – if student a failed one year there is a year when student b failed.
$T \models p_ap_b \subseteq p_ap_a$ – each year, student a passed their exams iff student b did.

(Repetition free) inclusion atoms

(I1-I4) Rules for repetition free inclusion [Casanova et.al. '84]

($I1$) $x \subseteq x$.

($I2$) If $x \subseteq z$ and $z \subseteq y$, then $x \subseteq y$.

($I3$) If $xyz \subseteq uvw$, then $xzy \subseteq uwv$.

($I4$) If $xy \subseteq uv$, then $x \subseteq u$.

(I1-I6) Rules for inclusion [cf. Mitchell '83]

($I5$) If $xy \subseteq uv$, then $xyy \subseteq uvv$.

($I6$) If $x_1x_2 \subseteq y_1y_1$ and $z \subseteq vx_2$, then $z \subseteq vx_1$.*

*Repetition on RHS allows us to express equalities between variables!

We give an alternative completeness proof using only two values.

Completeness theorem

Let Σ be a set of inclusion atoms. If $\Sigma \models x \subseteq y$, then $\Sigma \vdash_{\{I1-I6\}} x \subseteq y$.

Proof.

Assume that $\Sigma \nvdash x \subseteq y$. We define the counterexample team T by $s \in T$ iff

(1) $s : V \longrightarrow \{0, 1\}$,

(2) for z_i, z_j such that $\Sigma \vdash z_i z_j \subseteq z_k z_k$: $s(z_i) = s(z_j)$,

If there is y_j, y_k such that they are equal but x_j, x_k are not, then we stop here. Otherwise demand also:

(3) for w such that $\Sigma \vdash w \subseteq y$: $s(w) \neq 0^n$.

The value 0^n in condition (3) is never removed by (2), ensuring that $T \not\models x \subseteq y$. We can also show that for $u \subseteq v \in \Sigma$, $T \models u \subseteq v$. □

x_1	x_2	y_1	y_2	z_1
1	1	1	1	1
1	0	1	1	1
0	1	1	1	1
0	0	1	1	1
1	1	1	1	0
1	0	1	1	0
0	1	1	1	0
0	0	1	1	0

x_1	x_2	y_1	y_2	z_1
1	1	0	0	1
1	0	0	0	1
0	1	0	0	1
0	0	0	0	1
1	1	0	0	0
1	0	0	0	0
0	1	0	0	0
0	0	0	0	0

Table: Counterexample team consisting of the white lines, for the consequence $x_1x_2 \subseteq y_1y_2$ and assumption set Σ from which we can derive $y_1y_2 \subseteq z_1z_1$ and $x_1z_1 \subseteq y_1y_2$.

x_1	x_2	y_1	y_2	z_1
1	1	1	1	1
1	0	1	1	1
0	1	1	1	1
0	0	1	1	1
1	1	1	0	0
1	0	1	0	0
0	1	1	0	0
0	0	1	0	0

x_1	x_2	y_1	y_2	z_1
1	1	0	1	1
1	0	0	1	1
0	1	0	1	1
0	0	0	1	1
1	1	0	0	0
1	0	0	0	0
0	1	0	0	0
0	0	0	0	0

Table: Counterexample team consisting of the white lines, for the consequence $x_1x_2 \subseteq y_1y_2$ and assumption set Σ from which we can derive $y_2z_1 \subseteq z_1z_1$ and $x_1z_1 \subseteq y_1y_2$.

Since we only need two values to build counterexample teams, the system is complete for propositional inclusion atoms.

Propositional inclusion with constants

We allow $\top, \bot$ to appear in the atom, for which $s(\top) = 1$ and $s(\bot) = 0$ always hold.

Example

$T \models \top\bot \subseteq p_a p_b$ – there is a year where student a passed their exams but student b did not.
$T \models p_a \subseteq \top$ – student a passed their exams every year.

We add the following rules to $I1$-$I6$ from earlier:

($C1$) If $\top \subseteq \bot$, then $q \subseteq r$.
($C2$) If $p \subseteq q$, then $p\top \subseteq q\top$ and $p\bot \subseteq q\bot$. [Yang '22]

Let x be a sequence of constants $\top, \bot$.

($C3$) If $A \subseteq \{r \subseteq q \mid r_i \in \{p_i, \top, \bot\}\}$ is such that for any* x, there is $r \subseteq q \in A$ such that x matches the $\top\bot$-part in r, then $p \subseteq q$.

Consider the instances of $C3$ of the following form:

$$\text{If} \quad \begin{array}{c} \top p_2 \dots p_n \subseteq q \\ p_1 \top \dots p_n \subseteq q \\ \ddots \\ p_1 p_2 \dots \top \subseteq q \\ \bot\bot \dots \bot \subseteq q \end{array} \quad \text{then} \quad p \subseteq q. \qquad (*)$$

The number of assumptions, i.e., the arity, of $(*)$ for atoms of arity n is $n+1$, and we show that it cannot be reduced. Thus extended inclusion atoms do not have a k-ary proof system.

We consider the rule $(*)$ for atoms of arity n.

Theorem. There is no $k-$ary axiomatization of propositional inclusion with constants.

We follow the general strategy presented in Casanova et. al. '84:

(1) No assumption is derived from the others.

(2) The only nontrivial atom derivable from the assumption set is the conclusion of the rule.

Build teams witnessing (1).
For (2), let us first reduce the nontrivial atoms we need to consider.

Lemma

Let all $v \subseteq w \in \Sigma$ *be such that* w *never contains all of* q*, then* $\Sigma \not\models v \subseteq q$ *for nontrivial* $v \subseteq q$*.*

Proof.

We can build a team that satisfies all atoms in Σ but not $v \subseteq q$.
In particular, $\{p' \subseteq q' \mid p', q'$ proper subsequences of $p, q\} \not\models p \subseteq q$ due to the following team:

p_1	p_2	$\dots$	p_n	q_1	q_2	$\dots$	q_n
0	0	$\dots$	0	1	0	$\dots$	0
0	0	$\dots$	0	0	1	$\dots$	0
		$\ddots$				$\ddots$	
0	0	$\dots$	0	0	0	$\dots$	1

□

We must check all cases with different types of variable configurations of v in $v \subseteq q$.

The main cases:

- There is some r_j in v.
- $v_i \in \{p_1, \ldots, p_n, q_1, \ldots, q_n, \top, \bot\}$ for all $1 \leq i \leq n$,
  - There is some v_i such that $v_i = \bot$.
  - $v_i \in \{p_1, \ldots, p_n, q_1, \ldots, q_n, \top\}$ for all $1 \leq i \leq n$ with at least two occurrences of $\top$ or both something from q and $\top$ (or q and p).
  - $v_i \in \{p_1, \ldots, p_n, \top\}$ for all $1 \leq i \leq n$ with at most one $\top$ in v.
  - $v_i \in \{q_1, \ldots, q_n\}$ for all $1 \leq i \leq n$.

Let $v \subseteq q$ be such that $v_i \in \{p_1, \ldots, p_n, q_1, \ldots, q_n, \top, \bot\}$

(c) If there are between 1 and $n-1$ many $\bot$ in v, then we build the following team:

p_1	p_2	$\ldots$	p_n	q_1	q_2	$\ldots$	q_n
1	1	$\ldots$	1	1	1	$\ldots$	1
1	1	$\ldots$	1	0	0	$\ldots$	0

(g) If $v_i \in \{p_1, \ldots, p_n, \top\}$ for all $1 \leq i \leq n$, and there is ≥ 2 $\top$ in v, then construct:

p_1	p_2	$\ldots$	p_n	q_1	q_2	$\ldots$	q_n
0	0	$\ldots$	0	1	0	$\ldots$	0
0	0	$\ldots$	0	0	1	$\ldots$	0
		$\ddots$				$\ddots$	
0	0	$\ldots$	0	0	0	$\ldots$	1
0	0	$\ldots$	0	0	0	$\ldots$	0

By adjusting these two types of teams we can cover all cases.

Theorem. I1-I6 and C1-C3 form a complete system for propositional inclusion with constants: If $\Sigma \models p \subseteq q$, then $\Sigma \vdash p \subseteq q$.

The proof is similar to before, we first build a team that respects equalities and constants, and then find a value for condition (3) by $C3$.

Propositional and modal inclusion logic

In $CPL(\subseteq)$ and $ML(\subseteq)$, the inclusion atoms are of the form

$$\alpha_1 \dots \alpha_n \subseteq \beta_1 \dots \beta_n,$$

where α_i, β_i are in *CPL* (or *ML*). We obtain the same expressivity with atoms of the form

$$\top \subseteq \alpha.$$

$CPL(\subseteq)$ is axiomatized in Yang '22, and $ML(\subseteq)$ in Anttila, H. & Yang '25.

Conclusion

Syntax	Exclusion atom	Axiomatization?
w/o repetitions	$x \mid y$	Yes, Cas83
w/ repetitions	$x \mid y$	Yes, Cas83+H24
w/ rep., approximations	$x \mid_r y$	Yes, H24
propositional	$p \mid q$	??

Syntax	Inclusion atom	Axiomatization?
w/o repetitions	$x \subseteq y, p \subseteq q$	Yes, Cas84
w/ repetitions	$x \subseteq y, p \subseteq q$	Yes, Mit84
w/ rep., constants	$\top \subseteq p$	Yes but not finite
w/ rep., approximations	$x \subseteq_r y$	??

Conclusion

Logic	Axiomatization?
$\mathrm{CPL}(\subseteq)$	Yang22
$\mathrm{ML}(\subseteq)$	AHY25
$\mathrm{CPL}(\mid)$	In progress
$\mathrm{CPL}(\mid, \subseteq)$	In progress

Thank you!

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